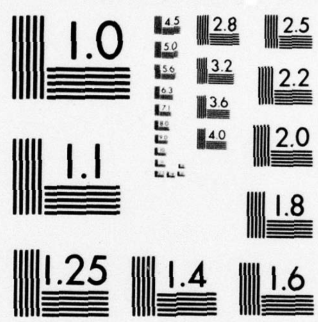
AD-A057 953 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 5/3
FAIR ALLOCATIONS OF A RENEWABLE RESOURCE.(U)
NOV 77 J SOBEL N00014-76-C-0134
UNCLASSIFIED ORC-77-31 NL

1 OF 1
AD
A057953

END
DATE
FILMED
10-78
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

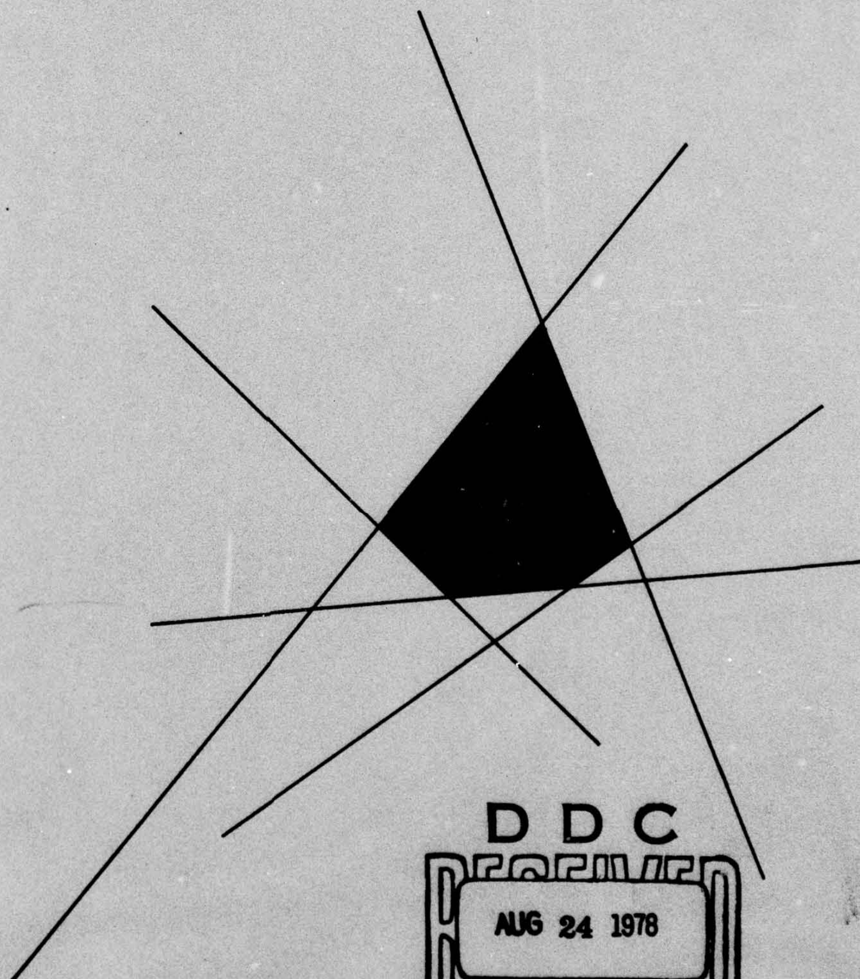
LEVEL II

ORC 77-31
NOVEMBER 1977

8

FAIR ALLOCATIONS OF A RENEWABLE RESOURCE

by
JOEL SOBEL



OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA • BERKELEY

FAIR ALLOCATIONS OF A RENEWABLE RESOURCE

by

Joel Sobel
Department of Mathematics
University of California, Berkeley

NOVEMBER 1977 ORC 77-31

This research has been partially supported by the Office of Naval Research under Contract N00014-76-C-0134 and the National Science Foundation under Grant MCS74-21222 A02 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

14

1. ~~REPORT NUMBER~~
ORC 77-31

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
6 FAIR ALLOCATIONS OF A RENEWABLE RESOURCE,

5. TYPE OF ~~REPORT & PERIOD COV~~ERED
9 Research Report,

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
10 Joel Sobel

8. CONTRACT OR GRANT NUMBER(s)
MCS74-21222 A02

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Operations Research Center
University of California
Berkeley, California 94720

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
National Science Foundation
1800 G Street
Washington, D.C. 20550

12. ~~REPORT DATE~~
11 Nov[redacted]977

13. ~~NUMBER OF PAGES~~
27

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
12 28 p.

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

15 N00014-76-C-0134, NSF-MCS74-21222

18. SUPPLEMENTARY NOTES

Also supported by the Office of Naval Research under Contract N00014-76-C-0134.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Growth Theory
Equity and Efficiency
Distributive Justice

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

270 750 alt

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ACKNOWLEDGMENT

I would like to take this opportunity to thank my advisor, David Gale. His advice and encouragement were indispensible throughout the preparation of this paper.

ABSTRACT

In an economy with finitely many agents, one renewable resource and an infinite horizon, it is shown that there is exactly one maximal allocation corresponding to given limiting shares of consumption and this allocation converges monotonically. Therefore, if there is no discounting, at most one fair maximal program exists - that which gives an equal amount to each individual in the limit. In this allocation, envy is always finite. However, only in special cases is it envy-free. This is in contrast to the case of finite economies where envy-free and Pareto efficient allocations may not exist, or if they exist may not be unique.

B

TABLE OF CONTENTS

INTRODUCTION

Mirman and Levhari [3] consider an infinite horizon economy with a single renewable resource. In [3], two countries fish in a common ocean. The fish population reproduces in accordance with the usual neoclassical production function. Each country has a utility function, and there is a discount rate common to both. It is shown that the Cournot-Nash non-cooperative duopoly equilibrium is in general not Pareto optimal.

In this paper, the cooperative solution for the same model with a finite number of agents is considered. We seek consumption programs which are maximal and satisfy some fairness criterion. The main result is that any maximal program is globally asymptotically stable in that *the value of capital stock* (fish population) monotonically approaches the "golden-rule" value $\bar{x}$ (that is, $f'(\bar{x}) = 1/\beta$ where f is the production function and β the discount rate) and the consumption of the i^{th} agent monotonically approaches some fixed value $\theta_i \bar{c}$, where $\theta_i \geq 0$, $\sum_{i=1}^{n} \theta_i = 1$ (here $\bar{c} = f(\bar{x}) - \bar{x}$ is the "golden rule" consumption, and n is the number of agents). Conversely there is exactly one maximal program corresponding to any distribution of limiting consumption. Fairness then consists in a reasonable choice of limiting consumptions. If the agents are thought of as individuals, equal limiting consumptions would seem appropriate. If they represent countries, the limiting shares could be chosen proportional to population. In this way, each individual could receive an equal limiting share of consumption. The allocations characterized by these definitions of fairness are not in general envy-free. However, in the undiscounted case $(\beta = 1)$ it is shown that our definition of fairness is the only one which guarantees that envy will be finite, that is, the utility an agent could receive from someone else's consumption stream can exceed the utility he actually receives by at most a finite amount.

The lack of an envy-free maximal allocation in the undiscounted case should not be objectionable. In fact, according to Rawls, "a rational individual does not suffer from envy. He is not ready to accept a loss for himself if only others have less as well .... Or at least this is true as long as the differences between himself and others do not exceed certain limits." ([6], Page 143.) Thus, we feel justified in asserting that the maximal allocation giving equal limiting consumptions to each individual is the only conceivable fair allocation in the case $\beta = 1$.

The existence of a unique fair efficient allocation in our model is in contrast to the case of finite economies. For example, any equilibrium for a pure exchange economy in which every agent is assigned an equal share of the initial resources is Pareto optimal and envy-free. (The resulting allocation is called income-fair in Pazner [4].) There is, however, no guarantee that there is a unique allocation having these properties. Moreover, in economies with production fair and efficient allocations need not exist. (An example is given in Pazner and Schmeidler [5].)

I. DEFINITIONS AND NOTATION

There are n agents. Each agent i has a utility function for consumption, $u_i : (0,\infty) \to \mathbb{R}$. u_i is assumed to be strictly increasing, strictly concave and twice continuously differentiable. Denote $\lim_{c\to 0} u_i(c)$ and $\lim_{c\to 0} u_i'(c)$ by $u_i(0)$ and $u_i'(0)$ respectively. These values need not be finite. A discount factor $\beta \in (0,1]$ is common to all consumers.

The technology is described by a twice continuously differentiable function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$. We shall assume, for all x , that $f'(x) > 0$ and $f''(x) < 0$. We shall also assume that $f'(x) > 1/\beta$ for some $x > 0$, and that $f(\hat{x}) = x$ for some $x > 0$. It then follows that there is a unique $\bar{x} \in (0,\hat{x})$ satisfying $f'(\bar{x}) = 1/\beta$. Let $\bar{c} = f(\bar{x}) - \bar{x}$. Notice $\bar{c} > \beta f(x) - x$ whenever $x \neq \bar{x}$.

A *program* is a sequence $\{(x_t;\gamma_t)\}_{t=1}^{\infty}$ with $x_t \geq 0$, $\gamma_t = \left(c_t^1, \ldots, c_t^n\right)$, $c_t^i \geq 0$ for all i and t . Let $c_t = \sum_{i=1}^{n} c_t^i$. A program $\{(x_t;\gamma_t)\}$ will be called *feasible* if, for some $x_0 > 0$, $c_t = f(x_{t-1}) - x_t$ for $t \geq 1$. We shall assume throughout that all programs start from a fixed $x_0 > 0$.

A sequence $\left\{\bar{c}_t^i\right\}_{t=1}^{\infty}$ is said to *catch up* to $\left\{c_t^i\right\}_{t=1}^{\infty}$ for agent i if $\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}\left[u_i\left(\bar{c}_t^i\right) - u_i\left(c_t^i\right)\right] \geq 0$. A program $\{(\bar{x}_t;\bar{\gamma}_t)\}$ is *maximal* if it is feasible, and for no other program $\{(x_t;\gamma_t)\}$ does $\left\{c_t^i\right\}$ catch up to $\left\{\bar{c}_t^i\right\}$ for each i . Notice that this definition coincides with the usual definition of Pareto optimality when $\beta < 1$. A feasible program $\{(x_t;\gamma_t)\}$ will be called *envy-free* if, for every i and j , there exists $T_0 > 0$ such that $\sum_{t=1}^{T} \beta^{t-1}\left[u_i\left(c_t^i\right) - u_i\left(c_t^j\right)\right] \geq 0$ whenever $T \geq T_0$.

II. UNIQUENESS OF MAXIMAL PROGRAMS

The purpose of this section is to show that for every distribution of limiting consumption there is at most one maximal program. To do this we shall restrict attention to programs satisfying a condition necessary for maximality. First, properties of feasible programs are deduced.

Let $x_m = \min(x_o, \bar{x})$. For $c \in (0, f(x_m) - x_m)$ define the function g_c by $g_c(y) = f(y) - c$. Since $g_c(\bar{x}) = f(\bar{x}) - c > f(\bar{x}) - (f(\bar{x}) - \bar{x})$ and $g_c(\hat{x}) = f(\hat{x}) - c \leq \hat{x}$, there is a $\hat{y} \in (\bar{x}, x]$ satisfying $g_c(\hat{y}) = \hat{y}$. Furthermore, $g_c(y) - y > 0$ for $y \in [x_m, \hat{y})$ and $g_c(y) - y < 0$ for $y > \hat{y}$.

<u>Lemma 2.1</u>:

The sequence defined by $y_t = g_c(y_{t-1})$, $y_0 = x_0$ converges monotonically to $\hat{y}$.

<u>Proof</u>:

If $y_t \in [x_m, \hat{y})$ then $\hat{y} = g_c(\hat{y}) > g_c(y_t) = y_{t+1} > y_t \geq x_m$. Therefore, since $y_0 \geq x_m$, $\hat{y} > y_0$ implies y_t increases to some $\tilde{y} \in (x_m, \hat{y}]$. Furthermore, $\tilde{y} = g_c(\tilde{y})$ so $\tilde{y} = \hat{y}$. If $y_o > \hat{y}$ a similar argument shows that y_t decreases to $\hat{y}$. ■

<u>Lemma 2.2</u>:

Let $x_M = \max(x_0, x)$. For any feasible program $\{(x_t; \gamma_t)\}$, x_t , $c_t^i \leq x_M$ for all i , and all $t \geq 1$.

Proof:

Let $y_t = f(y_{t-1})$, $y_0 = x_0$. By Lemma 2.1, y_t converges monotonically to $\hat{x}$, and hence $y_t \leq x_M$ for $t \geq 0$. Also, for any feasible program $x_t \leq y_t$ for $t \geq 0$. This follows by induction since $x_0 = y_0$ and if $x_s \leq y_s$ for some s , then $x_{s+1} = f(x_s) - c_{s+1}$ $\leq f(x_s) \leq f(y_s) = y_{s+1}$. Therefore, $x_t \leq y_t \leq x_M$ for $t \geq 0$. Since $c_t \leq f(x_{t-1})$ by feasibility, $c_t^i \leq c_t \leq f(x_{t-1}) \leq f(x_M) \leq x_M$ and the lemma is established. ■

Lemma 2.3:

Let $\{(x_t;\gamma_t)\}$ be a feasible program and let $c \in (0, f(x_m) - x_m)$. Suppose $x_t \leq \bar{x}$ for all $t \geq 1$. Then there exists an s such that $c_s \geq c$.

Proof:

Let $y_0 = x_0$, $y_t = g_c(y_{t-1})$. Pick T so that $y_T > \bar{x}$. This is possible by Lemma 2.1. Suppose $c_t \leq c$ for $t \geq 1$. Since $x_t = f(x_{t-1}) - c_t \geq f(x_{t-1}) - c$ it follows, by induction, that $x_t \geq y_t$ for $t \geq 0$. In particular, $x_T \geq y_T \geq \bar{x}$, contradicting $x_T \leq \bar{x}$ for $t \geq 1$. Hence $c_s > c$ for some s . ■

The following necessary condition for maximality makes it possible to restrict attention to programs $\{(x_t;\gamma_t)\}$ for which $\lim_{t\to\infty} (x_t;\gamma_t)$ exists.

Lemma 2.4:

Let $\{(x_t;\gamma_t)\}$ be a maximal program. Then, for every i and s,

(1) If $c_s^i > 0$ then $u_i'\left(c_s^i\right)/u_i'\left(c_{s+1}^i\right) \geqq \beta f'(x_s)$.

(2) If $c_{s+1}^i > 0$ then $u_i'\left(c_s^i\right)/u_i'\left(c_{s+1}^i\right) \leqq \beta f'(x_s)$.

In particular, if c_s^i, $c_{s+1}^i > 0$ then $u_i'\left(c_s^i\right)/u_i'\left(c_{s+1}^i\right) = \beta f'(x_s)$*.

Proof:

For fixed i and s define the function h_i by

$$h_i(\delta) = \beta^{s-1}\left[u_i\left(c_s^i - \delta\right) + \beta\, u_i\left(c_{s+1}^i + f(x_s+\delta) - f(x_s)\right)\right].$$

$h_i(\delta)$ is the utility agent i receives from consumption of $c_s^i - \delta$ in period s and $c_{s+1}^i + f(x_s+\delta) - f(x_s)$ in period $s+1$. Suppose $c_s^i > 0$, then we claim $h_i'(0) \leqq 0$. Otherwise $h_i(\delta) > h_i(0)$ for some $\delta \in \left(0, c_s^i\right)$. But then the program identical with $\{(x_t;\gamma_t)\}$ except that agent i consumes $c_s^i - \delta$ in period s and $c_{s+1}^i + f(x_s+\delta) - f(x_s)$ in period $s+1$ would dominate $\{(x_t;\delta_t)\}$.

*Lemma 2.4 makes it clear why a common discount rate is required. Suppose that agent i had a discount rate β_i for $i = 1,2$. If $\beta_1 > \beta_2$, $c_t^i > 0$ for $t \geqq T$, $i = 1,2$, then Lemma 2.4 implies $u_1'(c_T^1)/u_2'\left(c_T^2\right)(\beta_2/\beta_1)^N = u_1'\left(c_{T+N}^1\right)/u_2'\left(c_{T+N}^2\right)$ for all $N \geqq 0$. Hence $\lim_{t\to\infty} c_t^2 = 0$ and so no program satisfying conditions (1) and (2) of Lemma 2.4 can give positive consumption to both agents in the limit.

This contradicts the maximality of $\{(x_t;\gamma_t)\}$, so we may conclude that $c_s^i > 0$ implies $h_i'(0) \leq 0$. Similarly, if $c_{s+1}^i > 0$ then $h_i'(0) \geq 0$. The lemma follows since $h_i'(0) = -\beta^{s-1}\left[u_i'\left(c_s^i\right) - \beta f'(x_s)\, u_i'\left(c_{s+1}^i\right)\right]$.

The feasible program $\{(x_t;\gamma_t)\}$ will be called *admissible* if it satisfies conditions (1) and (2) above, for every i and t. ■

Proposition 2.5:

Let $\{(x_t;\gamma_t)\}$ be an admissible program. Then $\lim_{t\to\infty} (x_t;\gamma_t)$ exists and is equal to $(x;0)$ or $(x;\gamma)$ where $\gamma = (\bar{c}^1, \ldots, \bar{c}^n)$ and $\sum_{i=1}^{n} \bar{c}^i = \bar{c}$.

To prove this proposition it is necessary to prove the following.

Lemma 2.6:

Suppose $\{(x_t;\gamma_t)\}$ is an admissible program.

(1) If for some s, $x_{s-1} \geq x_s$, $\bar{x} \geq x_s$ then $x_t \geq x_{t+1}$ for $t \geq s$.

(2) If for some s, $x_{s-1} \leq x_s$, $\bar{x} \leq x_s$ then $x_t \leq x_{t+1}$ for $t > s$.

Thus, every admissible program is eventually monotone.

Proof:

It follows from Lemma 2.4 and the concavity of u_i that $x \geq x_s$ implies $c_{s+1} \geq c_s$ for each i. Therefore, $c_{s+1} = \sum_{i=1}^{n} c_{s+1}^i > \sum_{i=1}^{n}$

$c_s^i = c_s$, and so if $x_{s-1} \geq x_s$ then $x_{s+1} = f(x_s) - c_{s+1} \leq f(x_{s-1}) - c_s = x_s$. Hence, if $x_{s-1} \geq x_s$ and $\bar{x} \geq x_s$ for some s , then $\bar{x} \geq x_s \geq x_{s+1}$. Repeated applications of those reasoning establish (1). A symmetric argument yields (2). It follows that there exists an s such that either $x_t \geq \bar{x}$ for all $t \geq s$ or $x_t \leq \bar{x}$ for all $t \geq s$. Thus $\{\gamma_t\}$ is monotone for $t \geq s$.■

Proof of Proposition 2.5:

By Lemma 2.6, $\{(x_t;\gamma_t)\}$ is eventually monotone, therefore, by Lemma 2.2, $\lim_{t\to\infty} (x_t;\gamma_t)$ exists.

Let $\lim_{t\to\infty} (\bar{x}_t;\bar{\gamma}_t) = (\tilde{x};\tilde{\gamma})$. Suppose $\tilde{x} > \bar{x}$. Then we claim $\tilde{\gamma} = 0$. Otherwise $\lim_{t\to\infty} c_t^i = \tilde{c}^i > 0$ for some i . Pick T so that $\beta f'(x_t) \leq 1 - \delta$ for $t \geq T$ and some $\delta > 0$. Then $c_T^i \geq c_{T+N}^i \geq \tilde{c}^i$ for $N \geq 0$ and, by Lemma 2.4, $u_i'\left(c_T^i\right)/u_i'\left(c_{T+N}^i\right) = [\beta f'(x_T)] \ldots [\beta f'(x_{T+N-1})] \leq (1 - \delta)^N$. Thus $\lim_{N\to\infty} u_i'\left(c_T^i\right)/u_i'\left(c_{T+N}^i\right) = 0$. But this contradicts $u_i'\left(c_T^i\right)/u_i'\left(c_{T+N}^i\right) \geq u_i'(x_M)/u_i'(\tilde{c}^i) > 0$. Hence $\tilde{\gamma} = 0$ whenever $\tilde{x} > x$. Therefore, $0 = \lim_{t\to\infty} c_t = \lim_{t\to\infty} [f(x_{t-1}) - x_t] = f(\tilde{x}) - \tilde{x}$ and so $\tilde{x} = \hat{x}$.

To complete the proof it suffices to show $\tilde{x} < \bar{x}$ is impossible. In order to get a contradiction, assume $\tilde{x} < \bar{x}$. Let q be chosen so that $\beta f'(x_t) \geq 1 - \delta$ for all $t \geq q$ and some $\delta > 0$. If $x_t \leq \bar{x}$ for $t \geq 0$ then $\{\gamma_t\}$ is non-decreasing and, by Lemma 2.3, there exists $c > 0$ and a r such that $c_r > c$, and so $c_t \geq c$ for $t \geq r$. On the other hand, if $x_t > \bar{x}$ for some t , then there exists s such that $x_{s-1} > \bar{x} \geq x_s$ and $x_s \geq x_t$ for $t \geq s$. This is a consequence of Lemma 2.6. In this case, for $t \geq s$, $c_t \geq c_s = f(x_{s-1}) - x_s \geq f(\bar{x}) - \bar{x} = \bar{c}$. Now, let $T = \max (q,r,s)$

and $\varepsilon = \min(\bar{c}, c)/n$. Then, for some j , $c_t^j > \varepsilon$ whenever $t \geq T$. Hence $x_M \geq c_t^i$ for all i and t implies that $\infty > u_j'(\varepsilon)/u_j'(x_M)$ $\geq u_j'\left(c_T^j\right)/u_j'\left(c_{T+N}^j\right) = [\beta f'(x_T)] \ldots [\beta f'(x_{T+N-1})] \geq (1+\delta)^N$ for all $N \geq 0$. This is impossible, so $\hat{x} < \bar{x}$ is ruled out. The observation that if $\lim_{t\to\infty} x_t = \bar{x}$ then $\lim_{t\to\infty} c_t = f(\bar{x}) - \bar{x} = \bar{c}$ completes the proof.■

<u>Proposition 2.7:</u>

Suppose $\{(x_t;\gamma_t)\}$ is maximal. Then $\lim_{t\to\infty} x_t = \bar{x}$.

<u>Proof:</u>

Suppose the proposition is false. Let $\{(x_t;\gamma_t)\}$ be a maximal program such that $\lim_{t\to\infty} x_t \neq \bar{x}$. By Proposition 2.5 $\lim_{t\to\infty} x_t = \hat{x}$. Choose T so that $x_t \geq \bar{x}$, $|\hat{x} - x_t| < \bar{c}/4$, and $|\hat{x} - f(x_t)| < \bar{c}/4$ whenever $t \geq T$. It follows that $c_t = f(x_{t-1}) - x_t \leq |f(x_{t-1}) - \hat{x}| + |\hat{x} - x_t| < \bar{c}/2$. Consider the program $\{(\tilde{x}_t;\tilde{\gamma}_t)\}$ where

$$(\tilde{x}_t;\tilde{\gamma}_t) = (x_t;\gamma_t) \quad \text{if} \quad t < T$$

$$x_t = \bar{x} \quad \text{if} \quad t \geq T \quad \text{and}$$

$$c_t = \begin{cases} f(x_{T-1}) - \bar{x} & \text{if} \quad t = T \\ \bar{c} & \text{if} \quad t > T \end{cases} .$$

Since $c_t > c_t$ for $t > T$ we can choose $\tilde{\gamma}_t = \left(\tilde{c}_t^1, \ldots, \tilde{c}_t^n\right)$ such that $\sum_{i=1}^{n} c_t^i = c_t$ and $c_t^i > c_t^i$ for every i . Therefore $\{(\tilde{x}_t;\tilde{\gamma}_t)\}$ dominates $\{(x_t;\gamma_t)\}$, contradicting maximality.■

<u>Proposition 2.8:</u>

Suppose $\{(x_t;\gamma_t)\}$ is an admissible program such that $\lim_{t\to\infty} x_t = \bar{x}$. Then $\{(x_t;\gamma_t)\}$ is a monotone sequence, increasing if $x_0 < \bar{x}$, constant if $x_0 = \bar{x}$, and decreasing if $x_0 > \bar{x}$.

<u>Proof:</u>

The proposition follows immediately from Lemma 2.6 and Proposition 2.7.■

<u>Theorem 2.9:</u>

Given $\theta = (\theta_1, \ldots, \theta_n)$ such that $\theta_i > 0$ for each i, $\sum_{i=1}^{n} \theta_i = 1$ and $x_0 > 0$, there is at most one maximal program $\{(x_t;\gamma_t)\}$ starting from x_0 such that $\lim_{t\to\infty} \gamma_t = \bar{c}\theta$.

<u>Proof:</u>

Fix θ and let $M_{ij}(\theta) = M_{ij} = u_i'(\theta_i\bar{c})/u_j'(\theta_j\bar{c})$ for $1 \le i, j \le n$. Clearly $0 < M_{ij} < \infty$ and $M_{ij}M_{jk} = M_{ik}$.■

Before we can prove the theorem, two preliminary results are needed.

<u>Lemma 2.10:</u>

Let $\{(x_t;\gamma_t)\}$ be an admissible program with $\lim_{t\to\infty} (x_t;\gamma_t) = (\bar{x};\bar{c}\theta)$. Then

$$\beta f'(x_t) \ge u_j'\left(c_t^j\right)/u_j'\left(c_{t+1}^j\right)$$
$$\text{for all } j \text{ and } t \text{, with equality whenever } c_t^j > 0 \tag{*}$$

and

$$u_i'\left(c_t^i\right)/u_j'\left(c_t^j\right) \ge M_{ij} \quad \text{whenever } c_t^i > 0 \,. \tag{**}$$

Proof:

Unless $0 = c_t^j = c_{t+1}^j$, (*) follows from Proposition 2.8 and Lemma 2.4. If $0 = c_t^j = c_{t+1}^j$ then $x_t \leq \bar{x}$ and so $\beta f'(x_t) \geq 1 = u_j'(0)/u_j'(0)$. Therefore, if $c_t^i > 0$, $u_i'\left(c_t^i\right)/u_i'\left(c_{t+N}^i\right) = [\beta f'(x_t)] \ldots [\beta f'(x_{t+N-1})] \geq u_j'\left(c_t^j\right)/u_j'\left(c_{t+N}^j\right)$ for all $N \geq 1$. Hence, provided $c_t^i > 0$, $u_i'\left(c_t^i\right)/u_j'\left(c_t^j\right) \geq \lim_{N\to\infty} u_i'\left(c_{t+N}^i\right)/u_j'\left(c_{t+N}^j\right) = M_{ij}$.■

Next we deduce a technical result concerning solutions to (**).

Lemma 2.11:

Suppose $(c^1, \ldots, c^n)$ and $(\tilde{c}^1, \ldots, \tilde{c}^n)$ satisfy (**) . If $c^j > \tilde{c}^j$ for some j , then $c^i > \tilde{c}^i$ whenever $\tilde{c}^i > 0$.

Proof:

Suppose $c^j > \tilde{c}^j$ and $\tilde{c}^k > 0$. We have $u_j'(c^j)/u_k'(c^k) \geq M_{jk}$ and $u_k'(\tilde{c}^k)/u_j'(\tilde{c}^j) \geq M_{kj}$ hence $u_j'(c^j)/u_k'(c^k) \geq u_j'(\tilde{c}^j)/u_k'(\tilde{c}^k)$ and so $1 > u_j'(c^j)/u_j'(\tilde{c}^j) \geq u_k'(c^k)/u_k'(\tilde{c}^k)$ and thus $c^k > \tilde{c}^k$.■

To prove the theorem suppose $\{(x_t;\gamma_t)\}$ and $\{(\tilde{x}_t;\tilde{\gamma}_t)\}$ are two different programs such that $\lim_{t\to\infty} (x_t;\gamma_t) = \lim_{t\to\infty} (\tilde{x}_t;\tilde{\gamma}_t) = (\bar{x};\theta\bar{c})$. Then we can find $s \geq 1$ such that $(x_t;\gamma_t) = (\tilde{x}_t;\tilde{\gamma}_t)$ for $t \leq s$, and $\gamma_{s+1} \neq \tilde{\gamma}_{s+1}$. Without loss of generality, assume $\sum_{i=1}^n c_{s+1}^i > \sum_{i=1}^n \tilde{c}_{s+1}^i$. Then we claim that for all $t > s$, $x_t < \tilde{x}_t$ and $c_t^i > \tilde{c}_t^i$ whenever $c_t^i > 0$. Since $\sum_{i=1}^n c_{s+1}^i > \sum_{i=1}^n c_{s+1}^i$, $x_{s+1} = f(x_s) - \sum_{i=1}^n c_{s+1}^i < f(x_s) - \sum_{i=1}^n \tilde{c}_{s+1}^i = \tilde{x}_{s+1}$. But, by Lemma 2.10, γ_{s+1} and $\tilde{\gamma}_{s+1}$ satisfy (**) . It follows from Lemma 2.11 that $c_{s+1}^i > \tilde{c}_{s+1}^i$ whenever $\tilde{c}_{s+1}^i > 0$.

This establishes the claim when $t = s + 1$. Now suppose the claim is true for some $T > s$. By (*) and the assumption that $x_T < \tilde{x}_T$, we have $u_i'\left(c_T^i\right)/u_i'\left(c_{T+1}^i\right) = \beta f'(x_T) > \beta f'(\tilde{x}_T) \geq u_i'\left(\tilde{c}_T^i\right)/u_i'\left(\tilde{c}_{T+1}^i\right)$ provided $c_T^i > 0$. Hence, since $c_T^i \geq \tilde{c}_T^i$, $c_T^i > 0$ implies $\tilde{c}_{T+1}^i > c_{T+1}^i$. Moreover, by Proposition 2.8, $c_T^j > 0$ *for some* j, *so* $c_{T+1}^j > \tilde{c}_{T+1}^j$ and therefore, since $\tilde{\gamma}_{T+1}$ and $\tilde{\gamma}_{T+1}$ both must satisfy (**) it follows from Lemma 2.11 that $c_{T+1}^i > \tilde{c}_{T+1}^i$ whenever $\tilde{c}_{T+1}^i > 0$. Hence, $\sum_{i=1}^n c_{T+1}^i > \sum_{i=1}^n \tilde{c}_{T+1}^i$ and $x_{T+1} = f(x_T) - \sum_{i=1}^n c_{T+1}^i < f(\tilde{x}_T) - \sum_{i=1}^n \tilde{c}_{T+1}^i = \tilde{x}_{T+1}$. The claim then follows by induction.

Now select T so that for $t \geq T$, $\tilde{c}_t^1 > 0$. This is possible since $\lim_{t\to\infty} \tilde{c}_t^1 = \theta_1\bar{c} > 0$. From the claim, $c_t^1 > \tilde{c}_t^1$ for $t \geq T$. Therefore, $u_1'\left(c_{T+N}^1\right)/u_1'\left(\tilde{c}_{T+N}^1\right) = u_1'\left(c_T^1\right)/u_1'\left(\tilde{c}_T^1\right)[\beta f'(\tilde{x}_T)/\beta f'(x_T)] \ldots$

$[\beta f'(x_{T+N-1})/(\beta f'(x_{T+N-1})]$ for $N \geq 1$. But $x_t < \tilde{x}_t$ for $t \geq T$ so

$1 > u_1'\left(c_T^1\right)/u_1'\left(\tilde{c}_T^1\right) \geq u_1'\left(c_{T+N}^1\right)/u_1'\left(\tilde{c}_{T+N}^1\right)$ for all $N \geq 1$. Thus

$1 > \lim_{N\to\infty} u_1'\left(c_{T+N}^1\right)/u_1'\left(\tilde{c}_{T+N}^1\right) = u_1'(\theta_1\bar{c})/u_1'(\theta_1\bar{c}) = 1$. This is impossible, and the contradiction establishes the theorem. ■

III. EXISTENCE OF MAXIMAL PROGRAMS

The purpose of this section is to prove that maximal programs exist.

Theorem 3.1:

Let $\theta = (\theta_1, \ldots, \theta_n)$ with $\theta_i > 0$, $\sum_{i=1}^{n} \theta_i = 1$ be given. Then there exists a maximal program $\{(\bar{x}_t;\bar{\gamma}_t)\}$ such that $\lim_{t\to\infty} \bar{\gamma}_t = \theta\,\bar{c}$.

It turns out that this theorem is an easy consequence of the existence of maximal programs in economies with a single utility maximizing agent. Define the function U by $U(c) = \max \sum_{i=1}^{n} u_i(c_i)/u_i'(\theta_i\bar{c})$ subject to $c_i \geq 0$, $\sum_{i=1}^{n} c_i = c$. Clearly, U is continuous and strictly concave. Also, since each u_i is strictly concave, every $c \geq 0$ determines a unique vector $\gamma(c) = (c_1, \ldots, c_n)$ such that $c_i \geq 0$, $\sum_{i=1}^{n} c_i = c$ and $U(c) = \sum_{i=1}^{n} u_i(c_i)/u_i'(\theta_i\bar{c})$. We shall call $\{(x_t,c_t)\}$ a *feasible sequence* (from x_0) if $x_t \geq 0$, $c_t \geq 0$ and $c_t = f(x_{t-1}) - x_t$ for all $t \geq 1$. Thus, associated with every feasible sequence $\{(x_t;c_t)\}$ is a feasible program $\{(x_t;\gamma_t)\}$ where $\gamma_t = \gamma(c_t)$.

Theorem 3.1 is a consequence of the following result.

Proposition 3.2:

Given any $x_0 > 0$ there is a unique feasible sequence $\{(\bar{x}_t;\bar{c}_t)\}$ such that for any other feasible sequence $\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}[U(c_t) - U(\bar{c}_t)] < 0$.

Proposition 3.2 is well known in the theory of optimal growth. Proofs can be found (for the case $\beta = 1$), in more general settings, in Brock [1] or Gale [2]. Since only a weak version of their theorem is needed, an independent proof of Proposition 3.2 is given in the appendix.

To prove Theorem 3.1, let $\bar{\gamma}_t = \gamma(\bar{c}_t)$ for $t \geq 1$. Then Proposition 3.2 implies that $\{(x_t;\bar{\gamma}_t)\}$ is maximal. For suppose $\{(x_t;\gamma_t)\}$ is a feasible program such that $\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}\left[u_i\left(c_t^i\right) - u_i\left(\bar{c}_t^i\right)\right] \geq 0$ for each i. Then

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \beta^{t-1}[U(c_t) - U(\bar{c}_t)] =$$

$$\liminf_{T\to\infty} \sum_{t=1}^{T} \sum_{i=1}^{n} \frac{\beta^{t-1}}{u_i'(\theta_i\bar{c})}\left[u_i\left(c_t^i\right) - u_i\left(\bar{c}_t^i\right)\right] \geq$$

$$\sum_{i=1}^{n} \liminf_{T\to\infty} \sum_{t=1}^{T} \frac{\beta^{t-1}}{u_i'(\theta_i\bar{c})}\left[u_i'\left(c_t^i\right) - u_i\left(\bar{c}_t^i\right)\right] \geq 0 .$$

But, by Proposition 3.2, this can only happen if $\{(x_t;\gamma_t)\} = \{(\bar{x}_t;\bar{\gamma}_t)\}$ is maximal.

It remains to show that $\lim_{t\to\infty} \gamma_t = \theta\,\bar{c}$. Because $\{(\bar{x}_t;\bar{\gamma}_t)\}$ is maximal, it follows from Proposition 2.5 that $\lim_{t\to\infty} \gamma_t$ exists. Denote this limit by $(\bar{c}_1, \ldots, \bar{c}_n)$. But, by Propositions 2.5 and 2.7, $\sum_{i=1}^{n} \bar{c}_i = \bar{c}$ and so $(\bar{c}_1, \ldots, \bar{c}_n)$ solve: Maximize $\sum_{i=1}^{n} \frac{u_i(c_i)}{u_i'(\theta_i\bar{c})}$ subject to $\sum_{i=1}^{n} c_i = \bar{c}$, $c_i \geq 0$. Therefore, $u_i'(\bar{c}_i)/u_i'(\theta_i\bar{c}) \geq u_j'(\bar{c}_j)/u_j'(\theta_j\bar{c})$ whenever $\bar{c}_i > 0$. Thus $\sum_{i=1}^{n} \bar{c}_i = c$ and $\frac{u_i(\bar{c}_i)}{u_j'(\bar{c}_i)} \geq M_{ij}$ whenever $\bar{c}_i > 0$ and we must have $\bar{c}_i = \theta_i\bar{c}$ by Lemma 2.11. It follows that

$\lim_{t\to\infty} \gamma_t = \theta \bar{c}$, completing the proof. ∎

Combined with Theorem 2.9, Theorem 3.1 guarantees the existence of a unique maximal program associated with every distribution of limiting consumptions.

IV. FAIR ALLOCATIONS

The results of Sections II and III show that there is a unique maximal program corresponding to every limiting distribution of consumption. This section discusses the properties of maximal programs, with emphasis on that which gives equal shares to each agent in the limit. Throughout this part we assume $\beta = 1$.

We begin with a characterization valid for all maximal programs.

Proposition 4.1:

Let $\{(x_t;\gamma_t)\}$ be a feasible program starting from x_0 . Then, for all $T > 0$, $\sum_{t=1}^{T} (c_t - \bar{c}) < f(x_0)$.

Proof:

Since $c_t = f(x_{t-1}) - x_t$ and $f(x) - x \leq \bar{c}$ for all $x \geq 0$ we have $\sum_{t=1}^{T} (c_t - \bar{c}) = \sum_{t=1}^{T} [(f(x_t) - x_t) - \bar{c}] + f(x_0) - f(x_T) < f(x_0)$. ■

Lemma 4.2:

For every maximal program $\{(\bar{x}_t;\bar{\gamma}_t)\}$, $\sum_{t=1}^{\infty} |\bar{x}_t - \bar{x}|$ converges.

Proof:

If $x_0 < \bar{x}$ Lemma 2.7 guarantees that $x_t \in [x_0,\bar{x}]$ for $t \geq 1$. Hence, by Lemma 2.3, there is a T such that $\bar{c}_T > 0$ and so there exists $c > 0$ and j such that $\bar{c}_t^j > c$ for $t \geq T$. It follows from Lemma 2.4 that $u_j'(c)/u_j'(x_M) \geq u_j'\left(\bar{c}_t^j\right)/u_j'\left(\bar{c}_{t+N}^j\right) = f'(\bar{x}_t) \ldots f'(\bar{x}_{t+N-1})$ for all $N \geq 0$. Therefore, $u_j'(c)/u_j'(x_M) \geq \Pi_{t\geq T} f'(\bar{x}_t)$ and so $\Pi_{t=1}^{\infty} f'(\bar{x}_t) < \infty$. It is well known that this implies $\sum_{t=1}^{\infty} [f'(\bar{x}_t) - 1] < \infty$, hence, since $f'(\bar{x}) = 1$ we have $\infty > \sum_{t=1}^{\infty} [f'(\bar{x}_t) - f'(\bar{x})] =$

$\sum_{t=1}^{\infty} (\bar{x}_t - \bar{x}) f''(\xi_t)$ for some $\xi_t \in [\bar{x}_t, \bar{x}]$. Therefore, if $m = \min_{\xi \in [x_0, \bar{x}]} (-f''(\xi)) > 0$, $\infty > \sum_{t=1}^{\infty} (\bar{x}_t - \bar{x}) f''(\xi_t) \geq m \sum_{t=1}^{\infty} (\bar{x} - \bar{x}_t)$. A similar argument establishes the lemma when $x_0 > \bar{x}$. ■

Theorem 4.3:

For any maximal program $\{(\bar{x}_t; \bar{\gamma}_t)\}$, $\sum_{t=1}^{\infty} |\bar{c} - \bar{c}_t| < \infty$.

Proof:

If $x_0 \geq \bar{x}$ then by Proposition 2.8 $\bar{c}_t \geq \bar{c}$ for $t \geq 1$ and the result follows from Proposition 4.1.

If $x_0 < \bar{x}$, then $\bar{c} > \bar{c}_t$ for $t > 1$ and

$$
\begin{aligned}
\bar{c} - \bar{c}_t &= (f(\bar{x}) - \bar{x}) - (f(\bar{x}_{t-1}) - \bar{x}_t) \\
&= (f(\bar{x}) - f(\bar{x}_{t-1})) + (\bar{x}_t - \bar{x}_{t-1}) - (\bar{x} - \bar{x}_{t-1}) \\
&\leq (\bar{x} - \bar{x}_{t-1}) f'(x_0) + (\bar{x}_t - \bar{x}_{t-1}) - (\bar{x} - \bar{x}_{t-1}) \\
&\leq (\bar{x} - \bar{x}_{t-1})(f'(x_0) - 1) + (\bar{x}_t - \bar{x}_{t-1}) .
\end{aligned}
$$

Hence $\sum_{t=1}^{\infty} (\bar{c} - \bar{c}_t) \leq (f'(x_0) - 1) \sum_{t=1}^{\infty} (\bar{x} - \bar{x}_{t-1}) + \sum_{t=1}^{\infty} (\bar{x}_t - \bar{x}_{t-1})$.

This completes the proof since $\sum_{t=1}^{\infty} (\bar{x} - \bar{x}_{t-1}) < \infty$ by Lemma 4.2 and $\sum_{t=1}^{\infty} (x_t - x_{t-1}) = \bar{x} - \bar{x}_0$. ■

Together with Proposition 4.1, Theorem 4.3 says that no program can yield infinitely more consumption than a maximal program. Corollary 4.4 makes an analogous statement about utilities.

Corollary 4.4:

Suppose $\{(\bar{x}_t; \bar{\gamma}_t)\}$ is a maximal program. If $\lim_{t \to \infty} \bar{\gamma}_t = \bar{c}\,\theta$ for some $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_i > 0$ for each i and $\sum_{i=1}^{n} \theta_i = 1$, then $\sum_{t=1}^{\infty} |\bar{c}_t^i - \theta_i \bar{c}| < \infty$ and $\sum_{t=1}^{\infty} \left|u_i(\theta_i \bar{c}) - u_i\left(\bar{c}_t^i\right)\right| < \infty$.

Proof:

By Proposition 2.8, $|\bar{c}_t^i - \theta_i \bar{c}| \leq |\bar{c}_t - \bar{c}|$ for all i and $t \geq 1$. So $\sum_{t=1}^{\infty} |\bar{c}_t^i - \theta_i \bar{c}| < \infty$ by Theorem 4.2. Also $|u_i(\bar{c}_t^i) - u_i(\theta_i \bar{c})| = u_i'(\xi_t)[|\bar{c}_t^i - \theta_i \bar{c}|] \leq M_i[|\bar{c}_t - \bar{c}|]$ where $M_i = \max\left[u_i(\bar{c}_1^i), u_i'(\theta_i \bar{c})\right]$. The corollary now follows from another application of Theorem 4.2.

Corollary 4.5:

If $\theta_i = 1/n$ for each i then $\sum_{t=1}^{\infty} |\bar{c}_t^i - \bar{c}_t^j| < \infty$ and $\sum_{t=1}^{\infty} |u_i(\bar{c}_t^i) - u_i(\bar{c}_t^j)| < \infty$ for every i and j.

Proof:

Since $|\bar{c}_t^i - \bar{c}_t^j| \leq |\bar{c}_t^i - \bar{c}/n| + |\bar{c}_t^j - \bar{c}/n|$, that $\sum_{t=1}^{\infty} |\bar{c}_t^i - \bar{c}_t^j| < \infty$ follows from Theorem 4.3. Also, if $M_i = \max\left[u_i'(\bar{c}_1^i), u_i'(\bar{c}/n)\right]$ then $\sum_{t=1}^{\infty} |u_i(\bar{c}_t^i) - u_i(\bar{c}_t^j)| \leq M_i \sum_{t=1}^{\infty} |\bar{c}_t^i - \bar{c}_t^j| < \infty$ by the mean value theorem.

Corollary 4.5 guarantees that the maximal program giving each agent limiting consumption $\bar{c}/n$ is almost envy-free, in that each individual receives - up to a finite amount - as much utility from his consumption sequence as from that of anyone else. Clearly no other maximal allocation will have this property, with any other limiting consumptions there would be a T and an $\varepsilon > 0$ such that $\bar{c}_t^i > \bar{c}/n + \varepsilon > \bar{c}/n - \varepsilon > \bar{c}_t^j$ for some i and j and all $t \geq T$. Agent j would then prefer i's consumption to his own by an infinite amount.

Unless the agents have identical utility functions, no maximal allocation is envy-free. In fact, the following theorem implies that in many circumstances there exists i and j such that $\bar{c}_t^i > \bar{c}_t^j$ for all t.

Theorem 4.6:

Suppose $\{(\bar{x}_t;\bar{\gamma}_t)\}$ is a maximal program and $\lim_{t\to\infty} \bar{\gamma}_t = \bar{c}(1/n, \ldots, 1/n)$. If, for some i and j, $u_i = gou_j$ where g is twice continuously differentiable, increasing and concave then

1) If $x_0 < \bar{x}$ then $\bar{c}_t^i > \bar{c}_t^j$ whenever $\bar{c}_t^j > 0$.

2) If $x_0 > \bar{x}$ then $\bar{c}_t^j > \bar{c}_t^i$ for all $t \geq 1$.

Proof:

Suppose $x_0 < \bar{x}$. Then by Proposition 2.8 c_t^i, $c_t^j < \bar{c}/n$ for all $t \geq 1$. If, for some s, $\bar{c}_s^j > 0$ and $\bar{c}_s^j \geq \bar{c}_s^i$ then $u_j'(\bar{c}/n)/u_i'(\bar{c}/n) \leq u_j'\left(\bar{c}_s^j\right)/u_i'(\bar{c}_s^i)$ by Lemma 2.10 but $u_j'(\bar{c}/n)/u_i'(\bar{c}/n) = u_j'(\bar{c}/n)/g'(u_j(\bar{c}/n))u_j'(\bar{c}/n) = 1/g'(u_j(\bar{c}/n))$ and $u_j'\left(\bar{c}_s^j\right)/u_i'\left(\bar{c}_s^i\right) \leq u_j'\left(\bar{c}_s^i\right)/u'\left(\bar{c}_s^i\right) = 1/g'(u_j(\bar{c}/n))$. Hence $g'\left(u_j\left(\bar{c}_s^i\right)\right) \leq g'(u_j(\bar{c}/n))$ and thus $\bar{c}_s^i \geq \bar{c}/n$, by the concavity of g. This contradiction establishes (1). (2) follows from a similar argument and the observation that if $x_0 > \bar{x}$ then $c_t^i > 0$ for all $t \geq 1$.

It is easy to see that unless $u_i = au_j + b$ for some $a > 0$ and $u_i = gou_j$ for some twice continuously differentiable increasing function g, where either g or g^{-1} is strictly concave over some interval. Therefore, provided two agents have different preferences, there is a production function and an initial endowment that guarantees in the maximal program with equal limiting consumptions, an agent consumes more in every period than some other agent.

REFERENCES

[1] Brock, William, "On the Existence of Weakly Maximal Programs in a Multi-Sector Economy," Review of Economic Studies, Vol. 37, pp. 275-280, (1970).

[2] Gale, David, "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, Vol. 34, pp. 1-18, (1967).

[3] Levhari, David and Leonard J. Mirman, "The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution," University of Illinois, Urbana, (March 1977).

[4] Pazner, Elisha A., "Pitfalls in the Theory of Fairness," Journal of Economic Theory, Vol. 14, pp. 458-466, (1977).

[5] Pazner, Elisha and David Schmeidler, "A Difficulty in the Concept of Fairness," Review of Economic Studies, Vol. 41, pp. 441-443, (1974).

[6] Rawls, John, A THEORY OF JUSTICE, Harvard University Press, Cambridge, Massachusetts, (1971).

[7] Varian, Hal R., "Equity, Envy and Efficiency," Journal of Economic Theory, Vol. 9, pp. 63-91, (1974).

APPENDIX

In this appendix we give a proof of Proposition 3.1 for the case $\beta = 1$. When $\beta < 1$, the proposition follows easily from the fact that $\sum_{t=1}^{\infty} \beta^{t-1} U(c_t)$ converges for all feasible sequences.

Let $\bar{U} = \sum_{i=1}^{n} u_i(\theta_i \bar{c})/u_i'(\theta_i \bar{c})$ We need the following results:

A1. Associated with any feasible sequence $\{(x_t;c_t)\}$ is a sequence of non-negative numbers $\{\delta_t\}$ such that

$$\sum_{t=1}^{T} [U(c_t) - \bar{U}] = f(x_0) - f(x_T) - \sum_{t=1}^{T} \delta_t .$$

A2. There exists a feasible sequence $\{(x_t;c_t)\}$ such that

$$\sum_{t=1}^{T} [U(c_t) - \bar{U}] \geq M \quad \text{for some constant } M \text{ and } T \geq 1 .$$

For A1 let $\{(x_t;c_t)\}$ be a feasible sequence and let $U(c_t) = \sum_{i=1}^{n} u_i(c_t^i)/u_i'(\theta_i \bar{c})$ with $c_t^i \geq 0$, $\sum_{i=1}^{n} c_t^i = c_t$. It follows from the concavity of the u_i and the mean value theorem that $u_i(c_t^i) - u_i(\theta_i \bar{c}) \leq u_i'(\theta_i \bar{c})[c_t^i - \theta_i \bar{c}]$ for every i and t. Hence, since $U(c_t) - \bar{U} = \sum_{i=1}^{n} \frac{1}{u_i'(\theta_i \bar{c})} [u_i(c_t^i) - u_i(\theta_i \bar{c})]$, $U(c_t) - \bar{U} \leq c_t - \bar{c}$. Put $\delta_t = (c_t - \bar{c}) - (U(c_t) - \bar{U}) + [\bar{c} - (f(x_t) - x_t)]$. Then, since $\beta \leq 1$ and $\bar{c} \geq f(x_t) - x_t$, $\delta_t \geq 0$. Also, $U(c_t) - \bar{U} = f(x_{t-1}) - f(x_t) - \delta_t$ and so $\sum_{t=1}^{T} [U(c_t) - \bar{U}] = f(x_0) - f(x_T) - \sum_{t=1}^{T} \delta_t$ as desired.

To show A2 pick $\varepsilon \in (0, f(x_0) - x_0)$ and let $y_1 = f(x_0) - \varepsilon$, $y_t = f(y_{t-1}) - \varepsilon$ for $t \geq 2$. It follows from Lemma 2.1 that $y_s \geq \bar{x}$ for some s. Let

$$(x_t;c_t) = \begin{cases} (y_t;\varepsilon) & \text{for } t = 1, \ldots, s \\ (\bar{x}; f(x_s) - \bar{x}) & \text{for } t = s+1 . \\ (\bar{x};\bar{c}) & \text{for } t > s+1 \end{cases}$$

Then $\{(x_t;c_t)\}$ is a feasible sequence and $\sum_{t=1}^{T}[U(c_t) - \bar{U}] >$ $\sum_{t=1}^{s}[U(\varepsilon) - \bar{U}] = s[U(\varepsilon) - \bar{U}]$ and this yields A2.

We shall call the feasible sequence $\{(x_t;c_t)\}$ *good* if there exists M such that $\sum_{t=1}^{T}[U(c_t) - \bar{U}] > M$ for $T \geq 1$. Notice that 1 implies $\sum_{t=1}^{\infty}\delta_t$ converges for all good sequences. Thus $\lim_{t\to\infty}\delta_t =$ 0 and, since $\delta_t \geq \bar{c} - (f(x_t) - x_t)$, $\lim_{t\to\infty}(x_t;c_t) = (\bar{x};\bar{c})$ for all good sequences $\{(x_t;c_t)\}$.

Now let $\alpha = \inf\left\{\sum_{t=1}^{\infty}\delta_t : \delta_t \text{ corresponds to a good sequence}\right\}$. By A2, α is finite. Furthermore, there is a feasible sequence $\{(\bar{x}_t;\bar{c}_t)\}$ with $\sum_{t=1}^{\infty}\bar{\delta}_t = \alpha$. To see this take, for each N, a good sequence $\left\{\left(x_t^N;c_t^N\right)\right\}$ with $\sum_{t=1}^{\infty}\delta_t^N \leq \alpha + 1/N$. Since $\left(x_t^N;c_t^N\right)$ are bounded uniformly for all t and N (Lemma 2.2), there exists a subsequence $\{N_j\}$ such that $\lim_{j\to\infty}\left(x_t^{Nj};c_t^{Nj}\right)$ exists for each t. Call this limit $\{(\bar{x}_t;\bar{c}_t)\}$. It is easy to see that $\sum_{t=1}^{\infty}\bar{\delta}_t = \alpha$ and that $\{(\bar{x}_t;\bar{c}_t)\}$ is a feasible sequence.

Let $\{(x_t;c_t)\}$ be a feasible sequence. Then, by A1, $\sum_{t=1}^{T}[U(c_t) - U(\bar{c}_t)] = [f(\bar{x}_T) - f(x_T)] + \sum_{t=1}^{T}[\bar{\delta}_t - \delta_t]$. Since $\lim_{t\to\infty} f(x_t) = f(\bar{x})$ whenever $\sum_{t=1}^{\infty}\delta_t < \infty$, we have $\liminf_{T\to\infty}\sum_{t=1}^{T}[U(c_t) - U(\bar{c}_t)] \leq 0$.

To complete the proof we need to show that $\liminf_{T\to\infty}\sum_{t=1}^{T}[U(c_t) - U(\bar{c}_t)] < 0$ for any feasible sequence $\{(x_t;c_t)\}$ different from $\{(\bar{x}_t;\bar{c}_t)\}$. But this follows immediately from the strict concavity of U and f. ∎